PRIME POWER GROUPS IN WHICH EVERY COMMUTATOR OF PRIME ORDER IS INVARIANT*

BY

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In this paper I represent by G a group of order p^m (p an odd prime) and class k; by H_j the jth central of† G; and by G' the first cogredient of G. The term "ith commutator" is used in the sense explained in an earlier paper.‡

If the central of G' is cyclic, H_3 must contain an operation s that is not contained in H_2 whose pth power is contained in H_1, since otherwise the quotient group H_3/H_1, which is the second central of G', would contain only one subgroup of order p, and would therefore be cyclic.§ But this is impossible.‖ Also G must contain an operation, as A, such that $A^{-1}sA = st$, where t is contained in H_2, but not in H_1. Moreover if $s^{-1}ts = th$, h is invariant in G and $s^{-p}ts^p = th^p = t$, since s^p is invariant in G. Hence the order of h does not exceed p. It follows that¶ $(st)^p = s^p t^p h^{p(p-1)/2} = s^p t^p$. Hence $A^{-1}s^p A = (st)^p = s^p t^p$, and $t^p = 1$. This proves that *if G' has a cyclic central, H_2 must contain a commutator of order p that is not invariant in G.*

If now any cogredient of G, as the ith $(1 < i < k-1)$, contains a cyclic central, H_{i+1}/H_i must be cyclic; and it follows from what has just been proved that there must be contained in H_{i+1} a commutator t_{i+1} which is not contained in H_i, but whose pth power is contained in H_{i-1}. There must then be some operation of G, as A_{i+1}, such that

$$A_{i+1}^{-1}t_{i+1}A_{i+1} = t_{i+1}t_i,$$

where t_i is a second commutator contained in H_i, but not in H_{i-1}, and t_i^p is contained in H_{i-2}. If $i > 2$, there must be an operation A_i such that

$$A_i^{-1}t_iA_i = t_it_{i-1},$$

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† This notation is that of Burnside's *Theory of Groups of Finite Order*, first edition, p. 62, and not that of the second edition, p. 120. Cf. de Séguier, *Éléments de la théorie des groupes abstraits*, p. 87.

‡ Fite, these Transactions, vol. 7 (1906), p. 61.

§ Burnside, loc. cit., second edition, p. 131.

‖ Fite, these Transactions, vol. 15 (1914), p. 48.

¶ Fite, these Transactions, vol. 3 (1902), p. 337.

where t_{i-1} is a third commutator contained in H_{i-1}, but not in H_{i-2}, and t_{i-1}^p is contained in H_{i-3}. We can continue this argument until we arrive at an ith commutator of order p that is contained in H_2 but not in H_1. Hence *if any cogredient of G, as the ith, has a cyclic central, H_2 must contain an ith commutator of order p that is not invariant in G.*

If every ith $(i < k - 1)$ commutator of order p is invariant in G, as is the case, for example, when the ith commutator subgroup is cyclic, neither the ith cogredient nor any succeeding cogredient can have a cyclic central. Moreover there is no operation of H_{i+2} that is not contained in H_{i+1} whose pth power is contained in H_i. For if s were such an operation, there would be an operation, A, of G such that

$$A^{-1} sA = st_{i+1},$$

where t_{i+1} is a commutator contained in H_{i+1}, but not in H_i. It follows from this that

$$A^{-1} s^p A = s^p t_{i+1}^p t,$$

where t is contained in H_{i-1}, as may be seen by applying an argument similar to the one on page 134 to the corresponding operations of the quotient group G/H_{i-1}. Hence, since s^p is by hypothesis contained in H_i, t_{i+1}^p must be contained in H_{i-1}. But the preceding argument shows that this is possible only in case G contains a non-invariant ith commutator of order p.

Since every $(i + 1)$th commutator is also an ith commutator,* there can be no operation of H_{i+j} $(j > 2)$ that is not contained in H_{i+j-1} whose pth power is contained in H_{i+j-2}. Hence the order of every operation of G that is not contained in H_{k-1} must be at least p^{k-i}.

If we suppose that every ith commutator that is contained in H_2 and whose order does not exceed p^r $(r > 0)$ is invariant in G, we shall see that the facts just mentioned are particular phases of somewhat more general ones.

For if an operation s of H_{i+2} that is not contained in H_{i+1} were such that its p^rth power is contained in H_i, there would be an operation A of G such that

$$A^{-1} sA = st_{i+1},$$

where t_{i+1} is contained in H_{i+1}, but not in H_i. Moreover

$$A^{-1} s^{p^r} A = s^{p^r} t_{i+1}^{p^r} t,$$

where t is contained in H_{i-1}. Since s^{p^r} is contained in H_i, $t_{i+1}^{p^r}$ must be contained in H_{i-1}. If $i = 1$ this would be contrary to the hypothesis, since the order of the non-invariant first commutator t_{i+1} would be equal to, or less than, p^r. If $i > 1$, we can apply a similar argument to t_{i+1} and continue the process until we arrive at a non-invariant ith commutator that is contained

* Fite, these Transactions, vol. 7 (1906), p. 61.

in H_2 and whose order does not exceed p^r. But by hypothesis there is no such commutator. Hence *if every ith commutator of G that is contained in H_2 and whose order does not exceed p^r is invariant in G, the p^rth power of an operation of H_{i+2} that is not contained in H_{i+1} cannot be contained in H_i.*

It follows that the order of every operation of H_{i+j} $(j \geqq 2)$ that is not contained in H_{i+j-1} must be greater than $p^{r(j-1)}$, and that every operation not contained in H_{k-1} must be of order greater than $p^{r(k-i-1)}$. In particular, if every commutator whose order does not exceed p^r is invariant, the order of every operation that is not contained in H_{k-1} must be greater than $p^{r(k-2)}$.

The following statement follows from the preceding general discussion as an important special case: *If every operation of H_2 whose order does not exceed p^r is invariant in G, every operation of G whose order does not exceed p^r is invariant.*

If every ith commutator of order p that is contained in H_2 is invariant in G, no cogredient after the ith one can have a central that is generated by more independent generators than the central of the immediately preceding cogredient. This can be seen by comparing first the number of independent generators of the central of a group L of order a power of p with the number of independent generators of the central of the first cogredient of L. Let L_j and L' have the same significance with reference to L that H_j and G' respectively have with reference to G.

If the pth power of some operation of L_2 that is not contained in L_1 is equal to the pth power of some operation of L_1, there must be contained in L_2 an operation of order p that is not contained in L_1. If there is no such operation in L_2, the pth power of an operation of L_2 that corresponds to an operation of order p of L' must be some one of the $p^{a-n_1}(p^{n_1} - 1)$ operations of L_1 that are not pth powers of operations of L_1. Here p^a and n_1 represent respectively the order of L_1 and the number of its independent generators. If the central of L' is generated by n independent generators, L_2 contains $p^a(p^n - 1)$ operations that correspond to operations of order p of the central of L'. Moreover these operations can be divided into sets of p^{n_1} operations each such that the operations of any set have the same pth power, while two operations of different sets have different pth powers. This follows from the fact that L_1 contains p^{n_1} operations whose orders do not exceed p and the assumption that every operation of L_2 not contained in L_1 is of order greater than p. Hence L_1 must contain $p^{a-n_1}(p^n - 1)$ operations that are pth powers of operations of L_2, but not pth powers of operations of L_1. This would be impossible if $n > n_1$.

If now the central of the $(i + j)$th cogredient of $G\,(j \geqq 1)$—that is, the central of G/H_{i+j}—were generated by n independent generators, while the central of G/H_{i+j-1} is generated by n_1 independent generators, where $n > n_1$, we should have the case just considered with G/H_{i+j-1} for L. Hence the pth power of some operation of H_{i+j+1} not contained in H_{i+j} would be contained in H_{i+j-1}. But this is impossible since every ith commutator of order p that is contained in H_2 is invariant in G.

Suppose now that every commutator of G that is contained in H_2 and whose order does not exceed p^r is invariant in G, and that G contains an invariant commutator of this order, but none of higher order. Then the p^rth power of any operation of any central after the first one is contained in the preceding one. Moreover in any cogredient of G the commutators in the second central whose orders do not exceed the order of every invariant commutator of this cogredient must themselves be invariant in this cogredient.

If H_{k-1} is abelian let s be any one of its operations and A any operation of G not contained in H_{k-1}. Then

$$s^{-1}As = At,$$

where t is contained in, say, H_j, but not in H_{j-1}. Suppose that

$$A^{-1}tA = tt_1.$$

Then t_1 is contained in H_{j-1}. If now A', t', and t_1' are the operations of G/H_{j-2} that correspond respectively to A, t, and t_1, then

$$(A't')^{p^r} = A'^{p^r} t'^{p^r} t_1'^{p^r(p^r-1)/2},$$

since t_1' is commutative with both A' and t'. Hence

$$s^{-1}A^{p^r}s = A^{p^r}t^{p^r}t_1^{p^r(p^r-1)/2}t_2,$$

where t_2 is contained in H_{j-2}. But A^{p^r} is contained in H_{k-1} and is therefore commutative with s. Hence

$$t^{p^r}t_1^{p^r(p^r-1)/2}t_2 = 1.$$

This implies that $t^{p^r} = 1$ since $t_1^{p^r(p^r-1)/2}t_2$ is contained in H_{j-2} and t^{p^r} cannot be contained in H_{j-2} unless this is composed of the identity alone. Since then the order of the commutator t does not exceed p^r, t must be invariant in G. Hence no operation of H_{k-1} can give a non-invariant commutator and H_{k-1} must be contained in H_2. This requires that $k \leqq 3$.

Suppose now that H_{k-1} is of class $k_1 (> 1)$ and assume that $k \leqq 2j+1$ when H_{k-1} is of class j and $j < k_1$. We have just seen that this assumption is valid when H_{k-1} is abelian. Every (k_1-1)th commutator of H_{k-1} is invariant in H_{k-1}. If we take any such commutator for the s of the argument just used for the case $j = 1$, we shall see that this commutator must be contained in H_2. Hence the class of the corresponding central of G/H_2 is less than, or equal to,* $k_1 - 1$. But G/H_2 is of class $k - 2$, and therefore in accordance with our assumption $k - 2 \leqq 2(k_1 - 1) + 1$, or $k \leqq 2k_1 + 1$.

In general, if H_{k-l} $(l > 1)$ is abelian, let s be any one of its operations and A any operation of G not contained in H_{k-l}. Then

$$A^{-1} sA = st,$$

where t is contained in, say, H_j, but not in H_{j-1}. If

$$A^{-1} tA = tt_1$$

and

$$A^{-1} t_n A = t_n t_{n+1} \qquad (n = 1, 2, \cdots),$$

then

$$A^{-p^{lr}} sA^{p^{lr}} = st^{p^{lr}} t_1^{p^{lr}(p^{lr}-1)/2} \cdots t_n^x \cdots t_{n_1}^y,$$

when x equals the coefficient of the $(n + 2)$th term in the expansion of the p^{lr}th power of a binomial and $y = 1$ or the coefficient of the $(n_1 + 2)$th term in this expansion, according as $n_1 = p^{lr} - 1$ or as t_{n_1} is the first of this series of commutators with which A is commutative. Now H_{j-n} contains t_n and H_{j-l-1} contains

$$t_1^{p^{lr}(p^{lr}-1)/2} \cdots t_n^x \cdots t_{n_1}^y,$$

but not $t^{p^{lr}}$, unless H_{j-l} is composed of the identity alone. But $A^{p^{lr}}$ is commutative with s. Hence $t^{p^{lr}} = 1$. It follows from this that H_{k-l} must be contained in H_{l+1}, and $k - l \leqq l + 1$, or $k \leqq 2l + 1$. Suppose now that H_{k-l} is of class k_l (> 1) and assume that $k \leqq jl + j + l$ when H_{k-l} is of class j and $j < k_l$. We have just seen that this assumption is valid when H_{k-l} is abelian. Every $(k_l - 1)$th commutator of H_{k-l} is invariant in H_{k-l}, and can therefore give in G only commutators whose orders do not exceed p^{lr}. Every such $(k_l - 1)$th commutator must therefore be contained in H_{l+1}. Hence the class of the central of G/H_{l+1} that corresponds to H_{k-l} is less than, or equal to, $k_l - 1$. But G/H_{l+1} is of class $k - l - 1$, and therefore in accordance with our assumption $k - l - 1 \leqq (k_l - 1)l + k_l - 1 + l$, or $k \leqq k_l l + k_l + l$. Hence

If G is of order p^m (p an odd prime) class k and if every commutator of G that is contained in H_2 and whose order does not exceed the highest order of any invariant commutator is itself an invariant commutator, then H_{k-l} is of class k_l, where $k \leqq k_l l + k_l + l$.

Groups with cyclic commutator subgroups and of order p^m are included in the category considered here.

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* Fite, loc. cit.